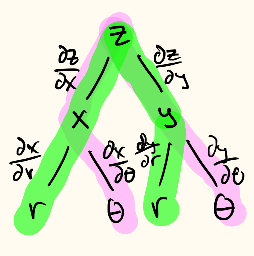
Meeting 9 Math 22

Reading Debrief

- Discuss Activities 10.5.2(c) & 10.5.3 w/ your group.
- Questions about the chain rule?
- Class discussion about Preview Activity 10.6.1.

10.5.3(b)

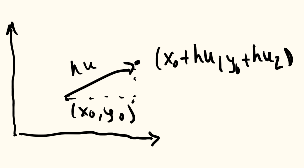


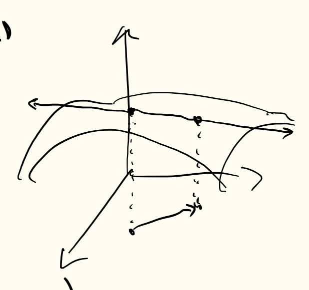
$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

Section 10.6.1 Directional Derivative

We want to measure the rate of change of a function $f(x,y)$ as we travel in the direction of a unit vector $u = \langle u_1, u_2\rangle$ at a point (x_0, y_0).
Suppose we travel a distance h in the direction of u.





The average rate of change of f in the direction of u

$$m = \frac{f(x_0+hu_1, y_0+hu_2) - f(x_0,y_0)}{h}$$

This is the slope of the secant line in the picture. To get the instantaneous rate of change, let $h \to 0$.

Definition 10.6.2 Let $f(x,y)$ be a function and $u = \langle u_1, u_2\rangle$ a unit vector. The directional derivative of f at (x_0,y_0) in the direction of u is the limit

$$D_u f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+u_1h, y_0+u_2h) - f(x_0,y_0)}{h}$$

whenever this limit exists.

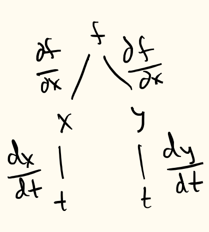
Section 10.6.2 Computing Directional Derivative

We can obtain a formula for D_uf by using the chain rule.
Parametrize the line through the point (x_0,y_0) in the direction of $u = \langle u_1, u_2\rangle$.

$$x(t) = x_0 + tu_1 \qquad y(t) = y_0 + tu_2.$$

Then

$$\begin{aligned}
D_uf(x_0,y_0) &= \lim_{h\to 0} \frac{f(x_0+hu_1, y_0+hu_2) - f(x_0,y_0)}{h}\\
&= \lim_{h\to 0} \frac{f(x(0+h), y(0+h)) - f(x(0),y(0))}{h}\\
&= \frac{d}{dt} f(x(t),y(t))\Big|_{t=0}\\
&= \frac{\partial f}{\partial x}(x(0),y(0))\frac{dx}{dt}\Big|_{t=0} + \frac{\partial f}{\partial y}(x(0),y(0))\frac{dy}{dt}\Big|_{t=0}\\
&= f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2\\
&= \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle \cdot u.
\end{aligned}$$



More compactly:

$$D_uf = \langle f_x, f_y\rangle \cdot u$$

Activity 10.6.2

- Complete Activity 10.6.2 and discuss w/ your group.
- Discuss as a class.

(a) $f_x(x,y) = 3y - 2xy^3$ $\qquad f_y(x,y) = 3x - 3x^2y^2$

(b) $\hat{i} = \langle 1, 0\rangle \qquad \hat{j} = \langle 0, 1\rangle$

$D_{\hat{i}}f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle 1, 0\rangle = f_x(x,y)$

$D_{\hat{j}}f(x,y) = \langle f_x, f_y\rangle \cdot \langle 0, 1\rangle = f_y(x,y)$

(c) Find a unit vector in the direction of v:

$$\frac{v}{|v|} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right\rangle$$

$$D_{\frac{v}{|v|}}f(1,-1) = \langle -1, 0\rangle \cdot \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right\rangle = -\frac{2}{\sqrt{13}}$$

Section 10.6.3 The Gradient

Definition Let $f(x,y)$ be a function. The gradient of f at (x_0,y_0) is the vector

$$\nabla f(x_0,y_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle.$$

The symbol ∇ is pronounced "del". With this new notation, the formula for the directional derivative is

$$D_uf(x_0,y_0) = \nabla f(x_0,y_0)\cdot u$$

or

$$D_uf = \nabla f \cdot u$$

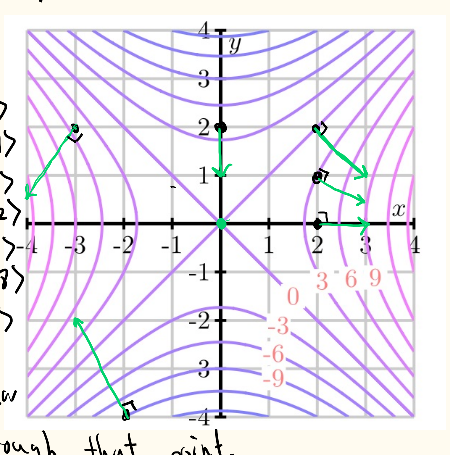
Activity 10.6.3

- Complete Activity 10.6.3 and discuss w/ your group.
- Class discussion. $\qquad f(x,y) = x^2 - y^2$

(a) $\nabla f(x,y) = \langle 2x, -2y\rangle$

(b)

- $(x_0, y_0) = (2, 0)$ $\quad \nabla f(2,0) = \langle 4, 0\rangle$
- $(x_0, y_0) = (0, 2)$ $\quad \nabla f(0,2) = \langle 0, -4\rangle$
- $(x_0, y_0) = (2, 2)$ $\quad \nabla f(2,2) = \langle 4, -4\rangle$
- $(x_0, y_0) = (2, 1)$ $\quad \nabla f(2,1) = \langle 4, -2\rangle$
- $(x_0, y_0) = (-3, 2)$ $\quad \nabla f(-3,2) = \langle -6, -4\rangle$
- $(x_0, y_0) = (-2, -4)$ $\quad \nabla f(-2,-4) = \langle -4, 8\rangle$
- $(x_0, y_0) = (0, 0)$ $\quad \nabla f(0,0) = \langle 0, 0\rangle$



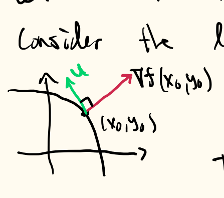
(c) $\nabla f(x_0,y_0)$ is perpendicular to the level curve through that point.

(d) It looks like the rate of change of f is positive in the direction of the gradient.

Section 10.6.4 Direction of the Gradient

The gradient is a vector. Let's see what we can say about which direction it points.

Let θ be the angle between $\nabla f(x_0,y_0)$ and u.
Consider the level curve through (x_0,y_0).



Suppose $\nabla f(x_0,y_0)$ and u are perpendicular.

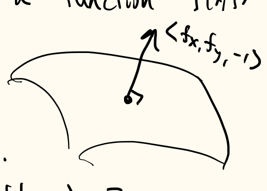
Then $D_uf(x_0,y_0) = \nabla f(x_0,y_0)\cdot u = 0.$

This means f is not changing in the direction of u. The only way this can happen is if u is tangent to the level curve. Thus, $\nabla f(x_0,y_0)$ is perp. to the level curve since it is perp. to u.

More generally, $\nabla f(x_0,y_0,z_0)$ is perpendicular to the level surface defined by the equation

$$f(x_0,y_0,z_0) = f(x,y,z).$$

We can use this fact to find an equation of the tangent plane to the graph of a function $f(x,y)$ at a point $(x_0, y_0, f(x_0,y_0))$.



Define a function

$$g(x,y,z) = f(x,y) - z.$$

The level surface $0 = g(x,y,z) = f(x,y) - z$ is just the graph of f. By the above the vector

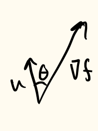
$$\nabla g(x,y,z) = \langle g_x, g_y, g_z\rangle = \langle f_x, f_y, -1\rangle$$

is normal to the graph of f. Thus, the equation of the tangent plane is

$$\langle f_x, f_y, -1\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0.$$

Activity 10.6.4

- Complete Activity 10.6.4 and discuss w/ your group. Skip part (a).
- Class discussion.



(a)

$$\begin{aligned}
D_uf(x_0,y_0) &= \nabla f(x_0,y_0)\cdot u\\
&= |\nabla f(x_0,y_0)|\cdot|u|\cos(\theta)\\
&= |\nabla f(x_0,y_0)|\cos(\theta)
\end{aligned}$$

(b) The maximum value of $\cos(\theta)$ is 1 and this happens when $\theta = 0$. Since $|\nabla f(x_0,x_0)|$ is fixed, this maximizes the directional derivative.

(c) They are parallel. The gradient points in the direction of steepest ascent.

(d) The direction of steepest descent is $-\nabla f(x_0,y_0)$.

(e) $u = \dfrac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|} \qquad v = -\dfrac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|} = -u.$

Section 10.6.5 Magnitude of the Gradient

The direction of steepest ascent is a unit vector u in the direction of the gradient $u = \dfrac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$.

What is the rate of increase of f in the direction of u? Compute the directional derivative:

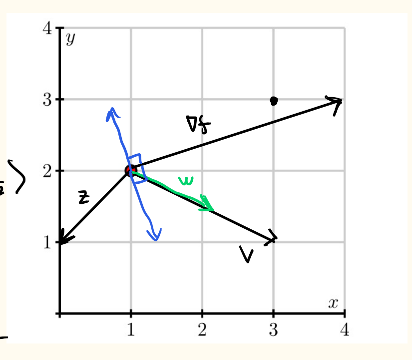
$$\begin{aligned}
D_uf(x_0,y_0) &= \nabla f(x_0,y_0)\cdot u\\
&= \nabla f(x_0,y_0)\cdot\frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}\\
&= \frac{|\nabla f(x_0,y_0)|^2}{|\nabla f(x_0,y_0)|} = |\nabla f(x_0,y_0)|
\end{aligned}$$

Activity 10.6.5

- Complete w/ your group.
- Class discussion.

(a) $\nabla f = \langle 2y-1, 2x-1\rangle$

$\nabla f(1,2) = \langle 3, 1\rangle$



(b) $D_zf(1,2) = \langle 3, 1\rangle \cdot \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle = -\frac{4}{\sqrt{2}} = -2\sqrt{2}$

(c) $D_zf(1,2)$ is the slope of the graph in the z direction. The sign tells us that we are decreasing.

(d) $w = \frac{v}{|v|} = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right\rangle$. D_wf is positive because the angle between ∇f and w is acute!

$$\begin{aligned}
D_wf(1,2) &= \nabla f(1,2)\cdot w\\
&= \langle 3, 1\rangle \cdot \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right\rangle = \frac{5}{\sqrt{5}} = \sqrt{5}.
\end{aligned}$$

(e) The direction of greatest ascent is

$$u = \frac{\nabla f(1,2)}{|\nabla f(1,2)|} = \frac{\langle 3, 1\rangle}{\sqrt{10}} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right\rangle$$

We have $D_uf(1,2) = |\nabla f(1,2)| = \sqrt{10}$.

(f). The direction of steepest descent is $\left\langle -\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\right\rangle$ the slope is $-\sqrt{10}$.

(g) The blue vectors.

(h) $\dfrac{\nabla f(3,3)}{\sqrt{50}} = \dfrac{\langle 5, 5\rangle}{\sqrt{50}} = \left\langle \frac{5}{\sqrt{50}}, \frac{5}{\sqrt{50}}\right\rangle$ points in the direction of steepest ascent.